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Comparison of numerical dispersion for finite-difference algorithms in transversely isotropic media with a vertical symmetry axis

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Abstract

Numerical simulation of the wave equation is widely used to synthesize seismograms theoretically and is also the basis of the reverse time migration and full waveform inversion. For the finite difference methods, grid dispersion often exists because of the discretization of the time and the spatial derivatives in the wave equation. How to suppress the grid dispersion is therefore a key problem for finite difference (FD) approaches. The FD operators for the space derivatives are usually obtained in the space domain. However, the wave equations are discretized in the time and space directions simultaneously. So it would be better to design the FD operators in the time–space domain. We improved the time–space domain method for obtaining the FD operators in an acoustic vertically transversely isotropic (VTI) media so as to cover a much wider range of frequencies. Dispersion analysis and seismic numerical simulation demonstrate the effectiveness of the proposed method.

Keywords: acoustic VTI modeling, time–space domain, finite difference scheme, dispersion relationship

(Some figures may appear in colour only in the online journal)

1. Introduction

Some hydrocarbon resource exploration and development projects are in vertically transversely isotropic (VTI) areas (Thomsen 1986, Du *et al* 2007). The acoustic VTI wave equations are commonly used and have proven to be useful in practice for better computational efficiency (Duveneck and Bakker 2011). Although many numerical methods have been investigated for wave equation simulation, the finite difference method is the most popular in seismic modeling research because of its high computational efficiency, smaller memory requirement and easy implementation (Lax and Wendroff 1964, Virieux 1984, Dablain 1986, Levander 1988, Moczo *et al* 2004, 2007, Liu and Sen 2009, 2010, Bielak *et al* 2010, Yan and Liu 2013). It also constitutes the basis for reverse time migration imaging and full waveform inversion (Kelly *et al* 1976, Dablain 1986, Yan and Liu 2013).

Grid dispersion is one of the key numerical problems affecting the practical usage when utilizing the finite difference method. The dispersion is due to the gridding in spatial and temporal partial derivatives of the acoustic wave equation. After grid discretization, the phase velocity becomes the function of the gridding intervals, which may cause the numerical phase velocity to be unequal to the true velocity of earth medium and hence reduce the precision of seismic wave simulations. Generally speaking, a finite difference scheme dominated by spatial dispersion decreases the phase velocities of high frequencies, while a finite difference scheme dominated by temporal dispersion increases the phase velocities of high frequencies (Dablain 1986).

Generally, the spatial FD coefficients are determined only in the spatial domain (Chu and Stoffa 2012). However, the wave equations are calculated in the temporal and spatial domain simultaneously (Finkelstein and Kastner 2006, 2008, Etgen 2007). Liu and Sen (2009) developed a new time–space domain FD method by using a plane wave theory and the Taylor series expansion. Zhang and Yao (2013a, 2013b) proposed to use the simulated annealing algorithm and give an error limitation for determining the FD coefficients in the space or the time–space domain. We proposed to satisfy the grid dispersion relationship in a range of frequencies and angles of propagation for the acoustic wave equation. In this paper, we extend the time–space domain method to the acoustic VTI wave equation.

2. Method

The exact phase velocity expression for VTI media is (Tsvankin 1996)

$$\frac{V^2(\theta)}{V_{p_0}^2} = 1 + \varepsilon\sin^2\theta - \frac{f}{2} \pm \frac{f}{2}\left(1 + \frac{2\varepsilon\sin^2\theta}{f}\right) \times \sqrt{1 - \frac{2(\varepsilon - \delta)\sin^2 2\theta}{f\left(1 + \frac{2\varepsilon\sin^2\theta}{f}\right)^2}} \quad (1)$$

where θ is the phase angle measured from the symmetry axis, V_{p_0} is the P wave velocity in the direction of symmetry axis, ε and δ are Thomsen's anisotropy parameters (Thomsen 1986) and $f = 1 - V_{s_0}^2 / V_{p_0}^2$ with the shear wave velocity along the symmetry axis denoted by V_{s_0}. Here, the plus sign corresponds to the P-wave and the minus sign corresponds to the SV-wave.

Setting V_{s_0} to zero in the elastic tensor for a VTI medium and using Hooke's law combining with the equations of motion, Duveneck's acoustic VTI wave equations with constant density are (Thomsen 1986, Duveneck and Bakker 2011, Yan and Liu 2013)

$$\begin{cases} \frac{1}{V_{p_0}^2}\frac{\partial^2\sigma_H}{\partial t^2} = (1 + 2\varepsilon)\frac{\partial^2\sigma_H}{\partial x^2} + \sqrt{1 + 2\delta}\frac{\partial^2\sigma_V}{\partial z^2} \\ \frac{1}{V_{p_0}^2}\frac{\partial^2\sigma_V}{\partial t^2} = \sqrt{1 + 2\delta}\frac{\partial^2\sigma_H}{\partial x^2} + \frac{\partial^2\sigma_V}{\partial z^2} \end{cases} \quad (2)$$

where σ_H, σ_V are the horizontal and vertical stress components, t is the time, x and z are the space coordinates.

The second order FD approximation for the second order time derivative is given by

$$\begin{cases} \frac{\partial^2\sigma_H}{\partial t^2} \approx \frac{\delta^2\sigma_H}{\delta t^2} = \frac{1}{\tau^2}\left(-2\sigma_H{}_{0,0}^{0} + \sigma_H{}_{0,0}^{1} + \sigma_H{}_{0,0}^{-1}\right) \\ \frac{\partial^2\sigma_V}{\partial t^2} \approx \frac{\delta^2\sigma_V}{\delta t^2} = \frac{1}{\tau^2}\left(-2\sigma_V{}_{0,0}^{0} + \sigma_V{}_{0,0}^{1} + \sigma_V{}_{0,0}^{-1}\right) \end{cases} \quad (3)$$

where τ is the time step, $\sigma_{m,n}^{j} = \sigma(x + mh, z + nh, t + j\tau)$ and h is the space grid space.

Utilizing the same coefficients for x and z directions, we have the difference scheme for the spatial derivatives:

$$\begin{cases} \frac{\partial^2\sigma_H}{\partial x^2} \approx \frac{\delta^2\sigma_H}{\delta x^2} = \frac{1}{h^2}\left(c_0\sigma_H{}_{0,0}^{0} + \sum_{m=1}^{M} c_m\left(\sigma_H{}_{-m,0}^{0} + \sigma_H{}_{m,0}^{0}\right)\right) \\ \frac{\partial^2\sigma_V}{\partial z^2} \approx \frac{\delta^2\sigma_V}{\delta z^2} = \frac{1}{h^2}\left(c_0\sigma_V{}_{0,0}^{0} + \sum_{m=1}^{M} c_m\left(\sigma_V{}_{0,-m}^{0} + \sigma_V{}_{0,m}^{0}\right)\right). \end{cases} \quad (4)$$

Using the plane wave theory and after some algebraic manipulations, Liu and Sen (2010) derived the following dispersion relationship equation

$$8r^{-2}\sin^2(\omega\tau) = -C + \sqrt{C^2 - 4D} \quad (5)$$

where

$$r = {V_{p_0}\tau}/{h} \quad (6a)$$

$$C = E + (1 + 2\varepsilon)F \quad (6b)$$

$$D = 2(\varepsilon - \delta)EF \quad (6c)$$

$$E = c_0 + 2\sum_{m=1}^{M} c_m\cos(mk_z h). \quad (6d)$$

$$F = c_0 + 2\sum_{m=1}^{M} c_m\cos(mk_x h). \quad (6e)$$

Therefore, the numerical phase velocity is

$$V_{FD} = \frac{\sin^{-1}\sqrt{{r^2}/{8} \times \left[-C + \sqrt{C^2 - 4D}\right]}}{k\tau}. \quad (7)$$

Taking a special angle of equation (5), Liu and Sen (2010) obtained

$$c_0 + 2\sum_{m=1}^{M} c_m\cos(mkh) = \frac{h^2}{V_{p_0}^2}\frac{2\cos(\varpi\tau) - 2}{\tau^2}. \quad (8)$$

When $\tau \to 0$, equation (8) becomes

$$c_0 + 2\sum_{m=1}^{M} c_m\cos(mkh) = -k^2h^2. \quad (9)$$

The conventional FD coefficient is determined by taking the Taylor expansion of the cosine function in equation (9) and then matching the term kh (Chu and Stoffa 2012). Using equation (9) as the objective function, the FD coefficients can also be obtained by using the least squares method or the simulating annealing method (Zhang and Yao 2013a, 2013b). Substituting $\varpi\tau = rkh$ into equation (8) and using the Taylor series expansion for the sine functions, the time–space domain dispersion-relation-based finite difference coefficients can be

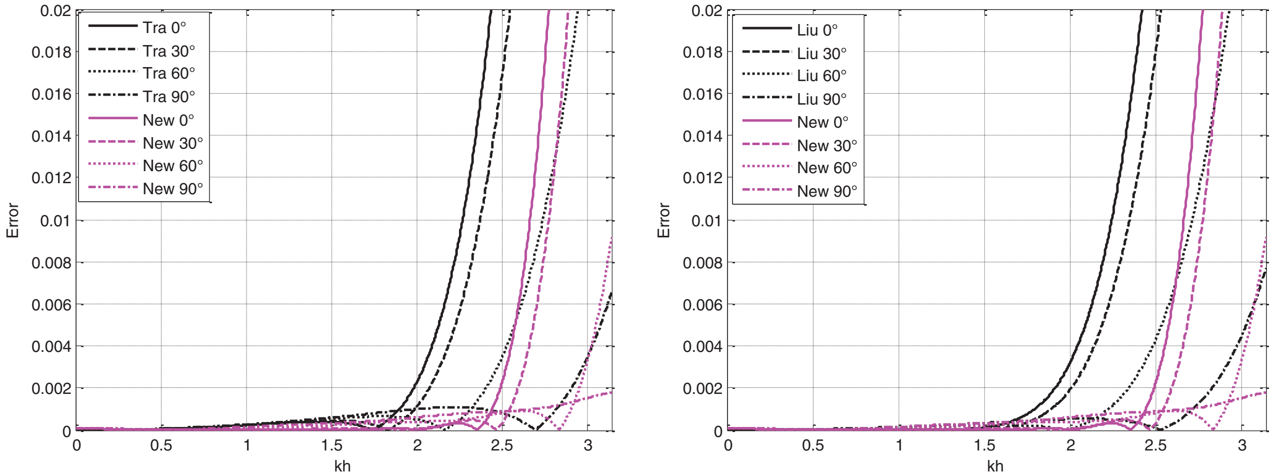

Figure 1. Dispersion errors of different methods where $v = 1500\,\mathrm{m\,s^{-1}}$, $\tau = 0.001$ s, $h = 20$ m, $M = 10$, $\varepsilon = 0.2$, $\delta = 0.05$ ($r = 0.075$).

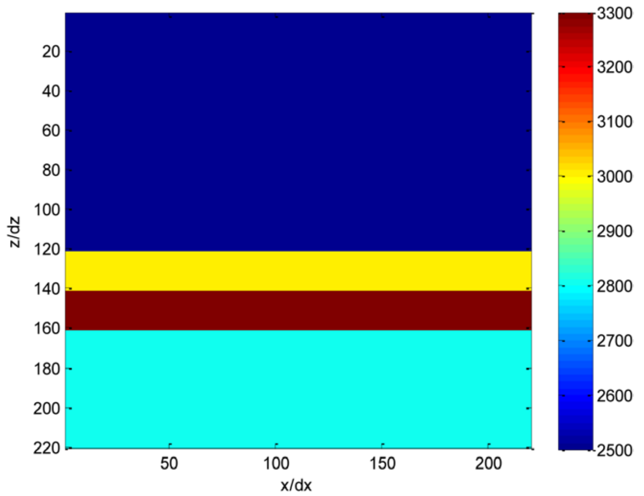

Figure 2. Parameters for the layered model are $v = 2500\,\mathrm{m\,s^{-1}}$, $\varepsilon = 0.1$, $\delta = 0.1$; $v = 3000\,\mathrm{m\,s^{-1}}$, $\varepsilon = 0.2$, $\delta = 0.05$; $v = 3300\,\mathrm{m\,s^{-1}}$, $\varepsilon = 0.15$, $\delta = 0.04$; $v = 2800\,\mathrm{m\,s^{-1}}$, $\varepsilon = 0.11$, $\delta = 0.06$ from top to bottom.

obtained (Liu and Sen 2010). However, with the Taylor expansion method in the time–space domain, the dispersion-relation are only preserved in the low wave numbers (frequencies).

We determine the upper limit of the wave numbers based on the maximum source frequency, the space grid interval and the wave velocity (Liang *et al* 2013)

$$\mathrm{ratio} = \frac{k_u}{k_{\mathrm{total}}} = \frac{2\pi f / v}{\pi / h} = \frac{f}{v / (2h)} \tag{10}$$

where f is related to maximum source frequency, v is seismic wave velocity and h is the spatial grid interval. The determination of ratio is important for the determination of the FD coefficients. If the ratio is too small, then the dispersion error in the high frequency would be high and the method is similar to the Taylor expansion method; if the ratio is too big, then the dispersion error in the low frequency would be high.

We assume that there are $M + 1$ grid points satisfying the dispersion relationship within the wave number range determined by the above method, then we could establish the following equations from equation (8):

$$\begin{pmatrix} 1 & 2\cos(k_{(1)}\Delta x) & \cdots & 2\cos(Mk_{(1)}\Delta x) \\ \vdots & \vdots & & \vdots \\ 1 & 2\cos(k_{(M+1)}\Delta x) & \cdots & 2\cos(Mk_{(M+1)}\Delta x) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_M \end{pmatrix} = \begin{pmatrix} \dfrac{2\cos(k_{(1)}v\tau) - 2}{r^2} \\ \vdots \\ \dfrac{2\cos(k_{(M+1)}v\tau) - 2}{r^2} \end{pmatrix} \tag{11}$$

where $k(i)$ $(i = 1, 2, \cdots, M + 1)$ is equally distributed between 0 and ratio π / h, where ratio is determined by equation (10). For the linear equation (11), either direct decomposition methods or iterative solvers could be applied for finding the FD coefficients. In the following, we will refer to the approach utilizing equations (11) as the linear method in the time–space domain.

3. Numerical dispersion analysis

We use the following formula

$$E = \left| \frac{V_{\mathrm{FD}} - V(\theta)}{V(\theta)} \right| \tag{12}$$

to measure the dispersion errors of the FD operator for the acoustic VTI wave equations, where the exact phase velocity $V(\theta)$ is defined in equation (1) with the plus sign and V_{FD} is defined as equation (7). Smaller values of E indicate lower dispersion of the FD operator.

Comparison of the dispersion errors using the Taylor expansion method in the space domain, the Taylor expansion and the linear method in the time–space domain is shown in figure 1. Figure 1 indicates that the time–space domain Taylor expansion method preserves the dispersion relationship over a larger kh range than the conventional method, while the time–space domain linear method preserves an even larger kh range than the time–space domain Taylor expansion method.

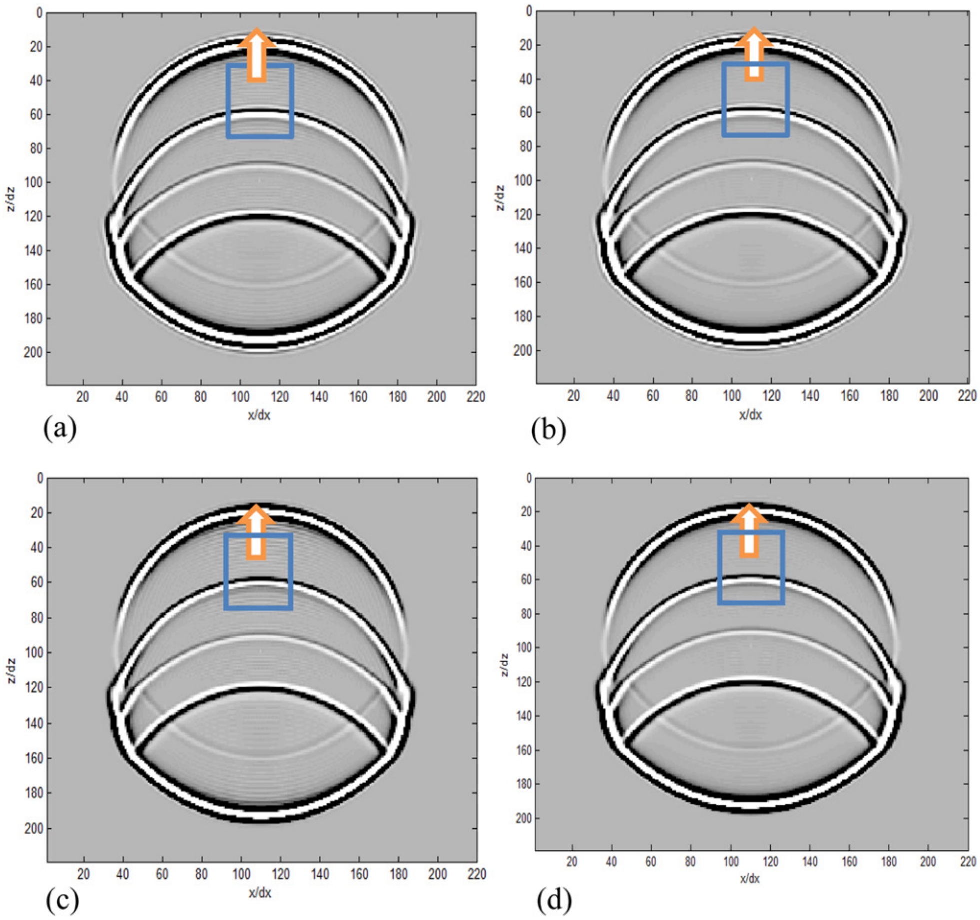

Figure 3. Snapshots of wave fields using different FD coefficients: (*a*) TE method in the space domain, (*b*) LS method in the space domain, (*c*) TE method in the time–space domain, and (*d*) the linear method in the time–space domain.

Table 1. The new FD coefficient used to obtain the snapshot in figure 3(*d*). The velocities are 2500, 3000, 3300, 2800 respectively from top to bottom, where $c_0 = -2\sum_{m=1}^{M} c_m$.

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
1.8587	−0.39796	0.14621	−0.062923	0.028048	−0.012095	0.0047611	−0.001592	0.00040009	−0.000056037
1.8275	−0.38247	0.13996	−0.060154	0.026798	−0.011552	0.0045464	−0.00152	0.00038198	−0.000053497
1.8063	−0.37197	0.13575	−0.05829	0.025956	−0.011187	0.004402	−0.00147	0.00036979	−0.000051789
1.8406	−0.38897	0.14258	−0.061314	0.027322	−0.01178	0.0046363	−0.00155	0.00038956	−0.00005456

4. Numerical examples

The source wavelet is the second derivative of the Gaussian function defined as

$$w(t) = c\exp\left(-f_0^2(t-t_0)^2\right) \qquad (13)$$

where the frequency f_0 is 150 Hz and f_0/π is the dominant frequency, t_0 is chosen as $4/f_0$ and c is a constant.

We first perform numerical simulation for a horizontally layered media. The numerical simulation parameters are shown as figure 2. The source function is applied at the center of the model in numerical simulating. The spatial sampling interval is 10 m, temporal step is 1 ms and M = 10 for all FD operators. The coefficients for figure 3(*d*) are shown in table 1.

We also note that Tan and Huang (2014) proposed to use an optimized method to get the FD coefficient with the full space

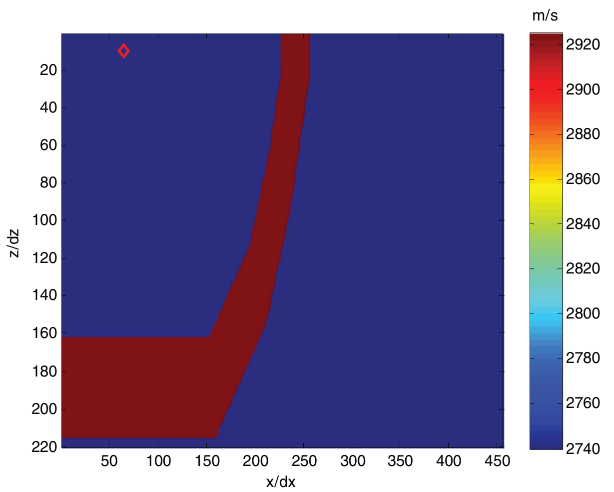

Figure 4. Velocity model.

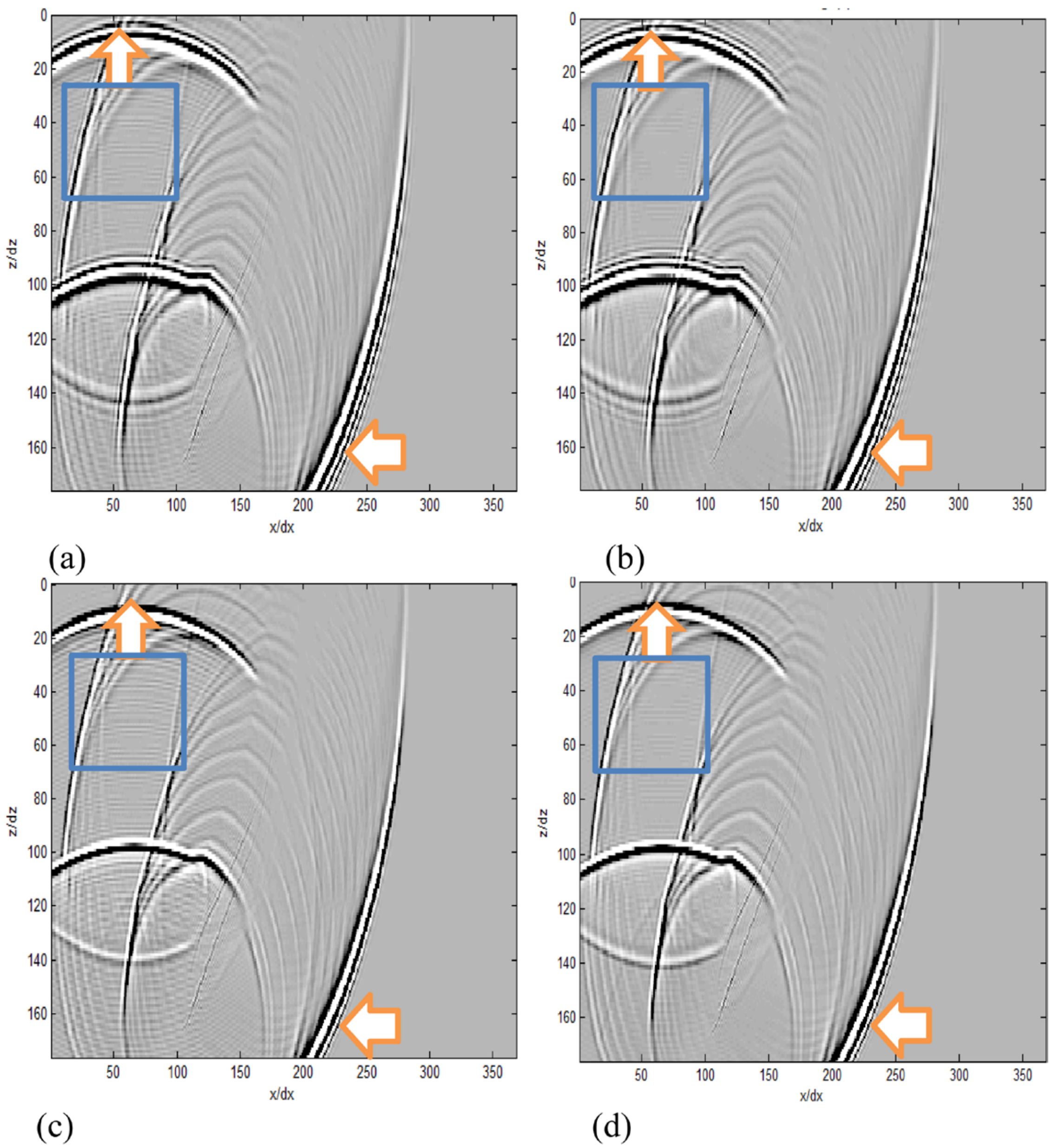

Figure 5. Snapshots of wave fields using different FD coefficients: (*a*) TE method in the space domain, (*b*) LS method in the space domain, (*c*) TE method in the time–space domain, and (*d*) the linear method in the time–space domain.

time discretization of the wave equation. With equation (5) as the object function, Tan and Huang's (2014) method may give a better result.

Figure 3 is the snapshots of the numerical simulation obtained from Taylor expansion (TE) method in the space domain, the least squares method (LS) in the space domain, Taylor expansion method in the time–space domain and the linear method in the time–space domain. The temporal dispersion is indicated by the arrow. The spatial dispersion is indicated by the rectangle.

From figure 3 we can observe that: both of the temporal and spatial dispersions are obvious when we use the TE method in the space domain; the spatial dispersion is reduced when we use the LS method in the space domain. Howerer, the temporal dispersion still exists; the temporal dispersion is reduced when we use the TE method in the time–space domain. However, the spatial dispersion still exists; both the temporal and spatial dispersion are reduced when we use the proposed time–space domain linear method.

The second example uses a velocity model shown as figure 4. The ε and δ are zeros when velocity is $2740\,\mathrm{m\,s^{-1}}$. When velocity is $2925\,\mathrm{m\,s^{-1}}$, ε equals 0.15 and δ equals 0.08 respectively.

The seismic source location is denoted with a red diamond. The source function is the same as the one used in the former example. The spatial sampling interval is 10 m, temporal step is 1 ms and $M = 10$ for all FD operators. We use different methods to determine the FD coefficients and then use these FD coefficients to simulate the acoustic VTI wave equations. The snapshots of different methods at 800 ms are shown as figure 5. The temporal dispersion is indicated by the arrow. The spatial dispersion is indicated by the rectangle. From figure 5 we clearly observe that the new method has effectively supressed the time and space grid dispersion as that in figure 3.

5. Conclusions

We compared the grid dispersion using the FD coefficients from different algorithms for VTI acoustic wave equation simulation. The proposed time–space domain linear method can reduce both the temporal and spatial dispersion effectively, while the previous method can only reduce the temporal or the spatial grid dispersion. At the same time, the amount of computation for calculating the FD coefficients using the linear method is small. So the time–space domain linear method should be used in VTI numerical simulation and reverse time migration for better accuracy and efficiency.

Acknowledgments

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