Optimization approximation with separable variables for the one-way wave operator

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[1] An optimization approximation with separable variables for the one-way wave operator is presented in this Letter. This new method approximates the one-way wave operator by products of functions in space variables and functions in wave number variables by means of optimization approximation with separable variables. This approximation enables us to use FFT algorithm which is independent of space variables while suffering no problem of branch points present in the generalized-screen method. *INDEX TERMS:* 0902 Exploration Geophysics: Computational methods, seismic; 0999 Exploration Geophysics: General or miscellaneous; 3230 Mathematical Geophysics: Numerical solutions. **Citation:** Chen, J.-B., and H. Liu (2004), Optimization approximation with separable variables for the one-way wave operator, *Geophys. Res. Lett.*, *31*, L06613, doi:10.1029/2004GL019429.

1. Introduction

[2] Consider the 3-D acoustic one-way wave equation for upcoming wave in the frequency-space domain

$$\frac{\partial U(z,x,y,\omega)}{\partial z} = i\sqrt{\frac{\omega^2}{c^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}\, U(z,x,y,\omega), \qquad (1)$$

where U is the wavefield, ω is the angular frequency, x is the lateral coordinate along the in-line direction, y is the lateral coordinate along the cross-line direction and $c = c(x, y, z)$ is the velocity function.

[3] Now we consider the one-way propagator associated with equation (1), i.e., the solution of equation (1) with initial condition

$$U(z = z', x, y, \omega) = \delta(x - x', y - y').$$

For a sufficiently small vertical step $\Delta z = z - z'$ (thin slab) and using the high-frequency approximation, the one-way thin slab propagator is given by *de Hoop et al.* [2000]:

$$g(z,x,y;z',x',y') \simeq \frac{1}{4\pi^2} \int \exp\left[i\sqrt{\frac{\omega^2}{c(\bar{z},x,y)^2} - \left(k_x^2 + k_y^2\right)}\,\Delta z\right]$$
$$\cdot \exp\left[i\left(k_x(x-x') + k_y(y-y')\right)\right] dk_x dk_y, \qquad (2)$$

where k_x, k_y are wave number and

$$\bar{z} = z' + \frac{1}{2}\Delta z.$$

For laterally homogeneous thin slab, i.e., $c(\bar{z}, x, y)$ is independent of x, y, the propagator reduces to Gazdag's phase-shift operator [*Gazdag*, 1978]. In this case, the computation of equation (2) requires only one two-dimensional FFT. For inhomogeneous thin slab, however, the computation of equation (2) requires one two-dimensional FFT for each different velocity $c(\bar{z}, x, y)$. This means a considerable computational effort. The split-step Fourier method introduced by *Stoffa et al.* [1990] requires much less computational cost by using a simple correction term applied in the ω, x domain to deal with lateral velocity variations. But this approach only works well for smooth velocity variations and near vertical propagation angles.

[4] *Le Rousseau and de Hoop* [2001] developed a scalar generalized-screen method which generalizes the phase-screen and the split-step Fourier methods to increase their accuracies with large and rapid lateral variations. Using two Taylor approximation and a perturbation hypothesis, this approach approximates the one-way wave operator by products of functions in space variables and functions in wave number variables. This approximation enables the dependency of equation (2) on x, y to be taken out of the integral thus resulting in a simplification of the computation. In spite of its great success, this method suffers a problem of branch points, and an integral contour deformation in the complex plane is needed.

[5] Recently, *Song* [2001] (an English translation of this paper can be obtained from chenjb@mail.igcas.ac.cn) suggested a theoretical method of expressing a multi-variable real function by products of single-variable functions. In fact, Song's method also works for complex functions. In this Letter, we will present a numerical implementation approach of Song's method for complex functions. Based on the numerical approach, we will obtain an approximation of the one-way wave operator which attains the goal of the generalized-screen method but suffers no problem of branch points.

[6] The new technique developed in this Letter can be used to construct fast 3-D wave-equation prestack depth migration algorithms. Now seismic imaging has been commonly applied to regions where geologic complexities are present. 3-D wave-equation prestack depth migration algorithms play a very important role in imaging regions with geologic complexities. However, because of the huge prestack data, the computational efficiency of the algorithms is in great demand. Therefore, the fast prestack depth migration algorithms based on this new technique

will be of great significance in imaging complex geologic regions.

2. Optimization Approximation With Separable Variables for the One-way Wave Operator

[7] We approximate the one-wave operator in the frequency-wave number domain

$$\mathcal{A}(u,k) = \exp\left(i\sqrt{u^2 - k^2}\,dz\right), \tag{3}$$

where $u = \frac{\omega}{c(x,y,z)}$, $k = \sqrt{k_x^2 + k_y^2}$.

[8] The optimization approximation with separable variables for equation (3) is to find functions $\phi(u)$, $\psi(k)$ and a complex number λ such that

$$\| \mathcal{A}(u,k) - \lambda\phi(u)\psi(k)^* \|_{L^2} = \min_{\tilde{\phi}, \tilde{\psi}, \tilde{\lambda}} \| \mathcal{A}(u,k) - \tilde{\lambda}\tilde{\phi}(u)\tilde{\psi}(k)^* \|_{L^2}, \tag{4}$$

where * denotes the complex conjugate, $\tilde{\lambda} \in \mathbb{C}$, and

$$\tilde{\phi} \in \left\{ \tilde{\phi}(u) : \tilde{\phi}(u) \in L^2[a,b], \| \tilde{\phi}(u) \|_{L^2} = 1 \right\},$$

$$\tilde{\psi} \in \left\{ \tilde{\psi}(k) : \tilde{\psi}(k) \in L^2[c,d], \| \tilde{\psi}(k) \|_{L^2} = 1 \right\}.$$

Using Lagrange multiplier, it can be easily proved that the solution to equation (4) is the eigenfunction corresponding to the eigenvalue with maximum modula of the following dual integral equation system

$$\begin{aligned} \int_c^d \mathcal{A}(u,k)\psi(k)dk &= \lambda\phi(u), \\ \int_a^b \mathcal{A}(u,k)^*\phi(u)du &= \lambda^*\psi(k). \end{aligned} \tag{5}$$

In general, the analytical solution of equation (5) is not available, and the system (5) can only be solved numerically. To obtain the numerical solution, we transform equation (5) into the following two independent self-adjoint integral equations

$$\int_a^b \int_c^d \mathcal{A}(u,k)\mathcal{A}(\tilde{u},k)^*\phi(\tilde{u})\,d\tilde{u}\,dk = |\lambda|^2\phi(u), \tag{6}$$

$$\int_a^b \int_c^d \mathcal{A}(u,k)^*\mathcal{A}(u,\tilde{k})\psi(\tilde{k})\,du\,d\tilde{k} = |\lambda|^2\psi(k). \tag{7}$$

Now we use two-dimensional numerical integration to solve equations (6) and (7). First consider the integral equation (6). Consider partitions of intervals $[a, b]$ and $[c, d]$ with nodes:

$$u_i = a + (i-1)\Delta u, \ i = 1, 2, \ldots, m+1; \quad \Delta u = \frac{b-a}{m},$$

$$k_j = c + (j-1)\Delta k, \ j = 1, 2, \ldots, n+1; \quad \Delta k = \frac{d-c}{n}.$$

Set $\phi(u_l) = \phi_l$, $l = 1, 2, \ldots, m + 1$. From equation (6), we have

$$\int_a^b \int_c^d \mathcal{A}(u_l,k)\mathcal{A}(\tilde{u},k)^*\phi(\tilde{u})\,d\tilde{u}\,dk = |\lambda|^2\phi_l, \quad l = 1, 2, \ldots, m. \tag{8}$$

We use repeated rectangle formula to integrate the left-hand side of equation (8), i.e, use rectangle formula on each rectangle $[u_i, u_{i+1}] \times [k_j, k_{j+1}]$, and obtain

$$\int_a^b \int_c^d \mathcal{A}(u_l,k)\mathcal{A}(\tilde{u},k)^*\phi(\tilde{u})\,d\tilde{u}\,dk$$

$$= \sum_{i=1}^m \sum_{j=1}^n \int_{u_i}^{u_{i+1}} \int_{k_j}^{k_{j+1}} \mathcal{A}(u_l,k)\mathcal{A}(\tilde{u},k)^*\phi(\tilde{u})\,d\tilde{u}\,dk$$

$$\approx \sum_{i=1}^m \left(\sum_{j=1}^n \Delta u\Delta k\mathcal{A}(u_l,k_j)\mathcal{A}(u_i,k_j)^* \right)\phi_i. \tag{9}$$

Set $\phi = (\phi_1, \phi_2, \ldots, \phi_m)^T$. Using equations (8) and (9), we obtain the equation satisfied by ϕ:

$$F\phi = |\lambda|^2\phi. \tag{10}$$

Here F is a matrix with entries:

$$f_{li} = \sum_{j=1}^n \Delta u\Delta k\mathcal{A}(u_l,k_j)\mathcal{A}(u_i,k_j)^*, \quad l, i = 1, 2, \ldots, m.$$

Further, let $A = (a_{i,j})$ be a matrix with entries:

$$a_{i,j} = \mathcal{A}(u_i,k_j), \quad i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n.$$

Then we have

$$F = AA^H, \tag{11}$$

where A^H denotes the conjugate transposition of A. For simplicity, we have incorporated the area factor of the rectangular into the eigenvalue and keep the notation for the eigenvalue unchanged.

[9] Now consider the integral equation (7). Set $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$. In the same way as before, we obtain the equation satisfied by ψ:

$$G\psi = |\lambda|^2\psi, \tag{12}$$

where

$$G = A^H A. \tag{13}$$

From equations (10)–(13), we can draw a conclusion that ϕ and ψ are the left and right singular vector of A corresponding to the maximum singular value λ_1 respectively. Using the power method (also called vector iteration method [*Stoer and Bulirsch*, 1993]), we can easily obtain ϕ from equation (10), and then we have $\psi = A^H\phi$. Let $\phi^{(1)}(u)$ and $\psi^{(1)}(k)$ denote the interpolation function of ϕ and ψ respectively.

[10] Now we obtain the optimization approximation with separable variables for $\mathcal{A}(u, k)$:

$$\mathcal{A}(u,k) \simeq \lambda_1\phi^{(1)}(u)\psi^{(1)}(k)^*.$$

To increase accuracy, set

$$\mathcal{A}_1(u,k) = \mathcal{A}(u,k) - \lambda_1\phi^{(1)}(u)\psi^{(1)}(k)^*.$$
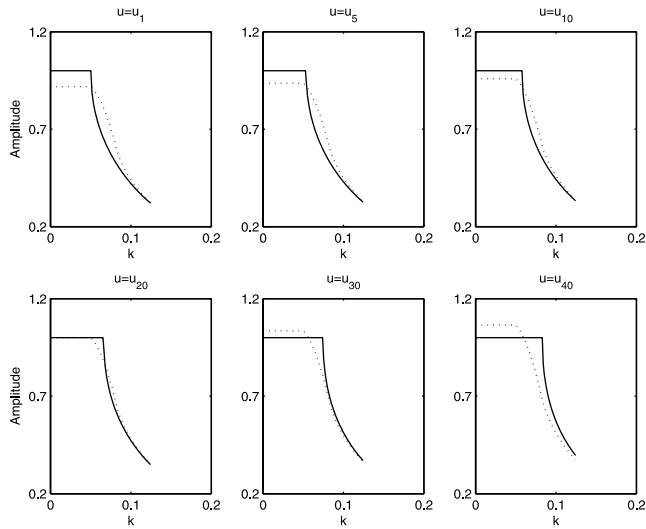
Figure 1. The first approximation of the one-way wave operator. The amplitude which varies with the wave number is shown.

We can obtain the optimization approximation with separable variables for $\mathcal{A}_1(u, k)$ by using the same method as used for $\mathcal{A}(u, k)$:

$$\mathcal{A}_1(u,k) \simeq \lambda_2\phi^{(2)}(u)\psi^{(2)}(k)^*.$$

Thus, we have the second order approximation

$$\mathcal{A}(u,k) \simeq \lambda_1\phi^{(1)}(u)\psi^{(1)}(k)^* + \lambda_2\phi^{(2)}(u)\psi^{(2)}(k)^*.$$

Repeating this process, we finally obtain

$$\mathcal{A}(u,k) \simeq \sum_{l=1}^{s}\lambda_l\phi^{(l)}(u)\psi^{(l)}(k)^*, \qquad (14)$$

where $s \leqslant r$ and r is the rank of A.

Figure 3. The convergence of (14). The amplitude which varies with the wave number is shown.

[11] The approximation (14) satisfies

$$\mathcal{A}(u_i,k_j) = \sum_{l=1}^{r}\lambda_l\phi^{(l)}(u_i)\psi^{(l)}(k_j)^*. \qquad (15)$$

Further, as $r \to \infty$, we have

$$\mathcal{A}(u,k) = \sum_{l=1}^{\infty}\lambda_l\phi^{(l)}(u)\psi^{(l)}(k)^* \qquad (16)$$

which converges at exponential rate [*Song*, 2001]. This expansion is closely related to the separable approximation of integral kernel [*Pipkin*, 1991].
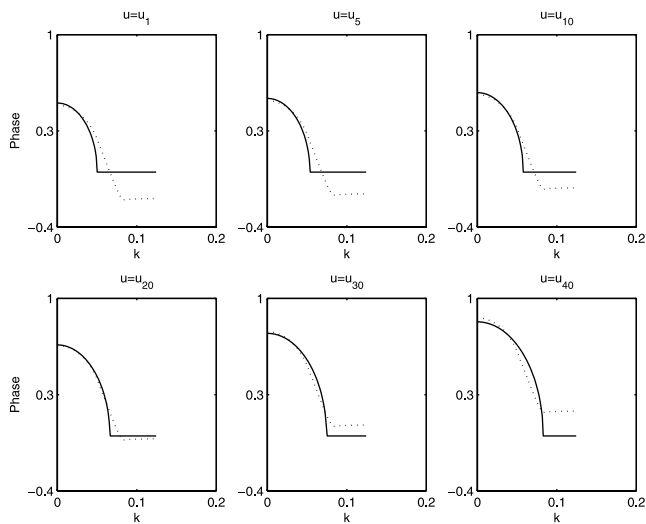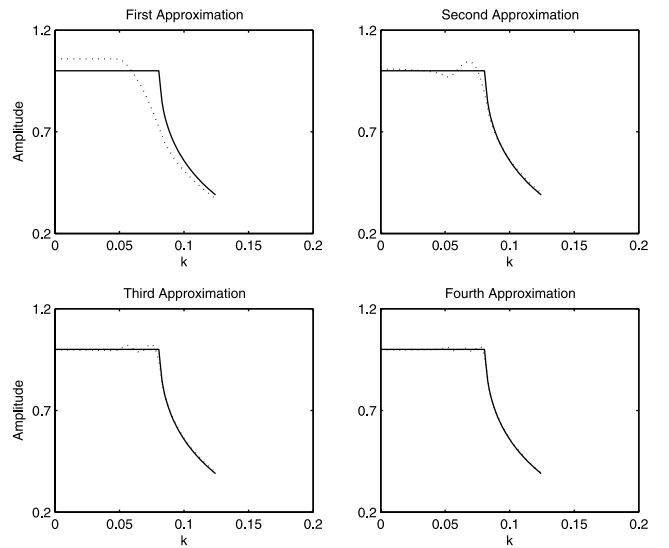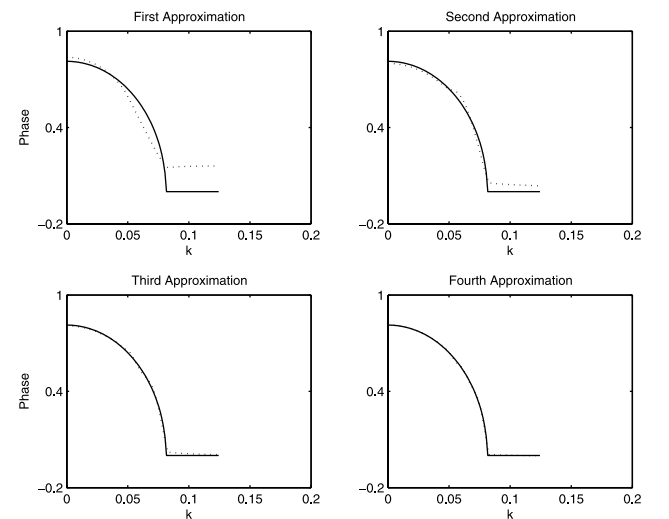
Figure 2. The first approximation of the one-way wave operator. The phase which varies with the wave number is shown.

Figure 4. The convergence of (14). The phase which varies with the wave number is shown.

Table 1. Errors for Different s

	$s = 1$	$s = 2$	$s = 3$	$s = 4$
Error	1.58439×10^{-2}	1.53274×10^{-3}	3.05616×10^{-4}	9.55636×10^{-5}

[12] Substituting equation (14) into equation (2), we obtain

$$g(z,x,y;z',x',y') \simeq \frac{1}{4\pi^2} \sum_{l=1}^{s} \lambda_l \phi^{(l)}(u) \int \psi^{(l)}(k)^* \\ \cdot \exp\left[i\left(k_x(x-x') + k_y(y-y')\right)\right] dk_x dk_y. \quad (17)$$

The computation of equation (17) requires s two-dimensional FFTs which are independent of u and therefor of x, y. Usually for a small number s, the approximation (14) can achieve very good accuracy. This means a great simplification of equation (2) by using equation (17). Furthermore, there is no problem of branch points in equation (17).

[13] **Remark.** In numerically solving equation (6), other high order numerical integration formulas can be chosen. The key point in choosing integration formulas is to insure that the resulting matrix F in equation (10) is an Hermitian matrix. For example, by direct calculations, it can be readily seen that applying the mid-point numerical integration formula results in an Hermitian matrix while applying the trapezoidal numerical integration formula does not lead to an Hermitian matrix.

3. Numerical Experiments

[14] Now we perform some numerical experiments. We take $a = \frac{40\pi}{2500}$, $b = \frac{40\pi}{1500}$ and $c = 0$, $d = \frac{\pi}{25}$. Set $m = 40$, $n = 100$. In Figures 1 and 2, we show the comparison between $\lambda_1 \phi^{(1)}(u)\psi^{(1)}(k)^*$ and $\mathcal{A}(u, k)$. We compare their amplitude (Figure 1) and phase (Figure 2) which vary with k for different u. We see that the first approximation $\lambda_1 \phi^{(1)}(u)\psi^{(1)}(k)^*$ preserves the basic shape of $\mathcal{A}(u, k)$. The accuracy varies with u. This just demonstrates that $\lambda_1 \phi^{(1)}(u)\psi^{(1)}(k)^*$ is the global approximation of $\mathcal{A}(u, k)$.

[15] In Figure 3, we show the approximation (14) for different s at u_{38} for the amplitude. Figure 4 shows the corresponding results for the phase. At accuracy of 10^{-6}, the iteration number of the power method is 3. Table 1 shows the following relative error for different s:

$$\text{Error} = \frac{1}{T} \sum_{i=1}^{100} \left| \mathcal{A}(u_{38}, k_i) - \sum_{l=1}^{s} \lambda_l \phi^{(l)}(u_{38})\psi^{(l)}(k_i)^* \right|^2, \quad (18)$$

where $T = \sum_{i=1}^{100} |\mathcal{A}(u_{38}, k_i)|^2$.

[16] For $s = 4$, the approximation (14) is already good enough and basically agrees with $\mathcal{A}(u, k)$. For different u, we can draw the same conclusion. Therefore, in the present example, the computation of equation (2) requires 40 FFTs while we only use 4 FFTs by using equation (17).

References

de Hoop, M. V., J. H. Le Rousseau, and R.-S. Wu (2000), Generalization of the phase-screen approximation for the scattering of acoustic waves, *Wave Motion*, 31, 43–70.

Gazdag, J. (1978), Wave equation migration with the phase-shift method, *Geophysics*, 43, 1342–1351.

Le Rousseau, J. H., and M. V. de Hoop (2001), Modelling and imaging with the scalar generalized-screen algorithms in isotropic media, *Geophysics*, 66, 1551–1568.

Pipkin, A. C. (1991), *A Course on Integral Equations*, Springer-Verlag, New York.

Song, J. (2001), The optimization expression of functions and manifolds in high dimensions by ones in low dimensions (in Chinese), *Chin. Sci. Bull.*, 46, 977–984. An English translation of this paper can be obtained at request to chenjb@mail.com.igcas.ac.cn.

Stoer, J., and R. Bulirsch (1993), *Introduction to Numerical Analysis*, 2nd ed., Springer-Verlag, New York.

Stoffa, P. L., J. T. Fokkema, R. M. de Luna Freir, and W. P. Kessinger (1990), Split-step Fourier migration, *Geophysics*, 55, 410–421.

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