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Short Note

Two kinds of separable approximations for the one-way wave operator

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INTRODUCTION

Le Rousseau and de Hoop (2001) developed a generalized screen method that generalizes the phase-screen and the split-step Fourier methods to increase their accuracies with large and rapid lateral variations. Using two Taylor approximations and a perturbation hypothesis, this approach approximates the one-way wave operator by products of functions in space variables and functions in wavenumber variables. This approximation enables the inverse Fourier transform with respect to wavenumbers to be independent of the space variables, thus resulting in significant improvement of the computational efficiency. In spite of its great success, this method has low convergence, and it suffers from the presence of branch points resulting from the choice of the background medium.

Chen and Liu (2004), without background medium, developed a method for constructing optimal approximations with separable variables for the one-way operator by developing the ideas in Song (2001). This new method approximates the one-way operator by products of functions in space variables and functions in wavenumber variables by means of the optimal approximation with separable variables. The approach presented in Chen and Liu (2004) has high convergence and does not suffer from the presence of branch points but needs numerical computations to determine the approximation.

In this short note, we will summarize the essential features of the above-mentioned methods and introduce the concept of separable approximations for the one-way wave operator. Two kinds of separable approximations based on the local and global frameworks are compared.

SEPARABLE APPROXIMATION FOR THE ONE-WAY OPERATOR

First consider the one-way thin slab propagator:

$$g(z,x,y;z',x',y') \simeq \frac{1}{4\pi^2}\int \exp\left[i\sqrt{\frac{\omega^2}{c(\bar{z},x,y)^2}-\left(k_x^2+k_y^2\right)}\,\Delta z\right] \times \exp\left[i\left(k_x(x-x')+k_y(y-y')\right)\right] dk_x dk_y, \quad (1)$$

where k_x, k_y are wavenumbers and

$$\bar{z} = z' + \frac{1}{2}\Delta z.$$

For details, see le Rousseau and de Hoop (2001):

The one-way operator in equation 1 is written as

$$\mathcal{A}(x,y;k_x,k_y) = \exp\left[i\sqrt{\frac{\omega^2}{c(x,y,\bar{z})^2}-\left(k_x^2+k_y^2\right)}\,\Delta z\right]. \quad (2)$$

The separable approximation for the one-way operator (2) has the following form:

$$\mathcal{A}(x,y;k_x,k_y) \sim \sum_{i=1}^{s} f_i(x,y) g_i(k_x,k_y). \quad (3)$$

With the approximation (3), the propagator (1) becomes

$$g(z,x,y;z',x',y') \simeq \frac{1}{4\pi^2}\sum_{i=1}^{s} f_i(x,y)\int g_i(k_x,k_y)\exp[i(k_x(x-x') + k_y(y-y'))]dk_x dk_y. \quad (4)$$

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Using equation 1 to compute the propagator requires a 2D inverse FFT (fast Fourier transform) for each space point in each depth interval because the FFT depends on the space variables. If we use equation 4, then the FFT no longer depends on the space variables and leads to a significant simplification of computational effort. The computational complexity of equation 4 for each depth interval is proportional to $(s+1)N_x N_y log_2(N_x N_y)$, where s refers to s inverse FFTs, 1 means one forward FFT, and N_x and N_y are the number of samples in x- and y-directions, respectively. The good performance of the separable approximation has been demonstrated on migration examples in le Rousseau and de Hoop (2001). The construction of the separable approximation (3) can be performed in two frameworks; one is local, the other global.

Local framework for constructing separable approximations

The local framework for constructing separable approximations consists of using the local Taylor expansion of the one-way operator (2) and assuming a reference velocity as a background velocity. Some methods impose no restrictions on the reference velocity, such as the split-step Fourier method (Stoffa et al., 1990) and the phase-screen method (le Rousseau and de Hoop, 2001). Some methods however, (e.g., the generalized screen method) (le Rousseau and de Hoop, 2001) impose conditions on the reference velocity, such as requiring that the reference velocity be smaller than the minimum velocity to avoid the branch points.

Let $c_0(\bar{z})$ denote the reference velocity in the depth interval under consideration. The perturbation $\Delta c(x, y, \bar{z})$ is given by

$$\Delta c(x, y, \bar{z}) = \frac{1}{c^2(x, y, \bar{z})} - \frac{1}{c_0^2(\bar{z})}.$$

In the following, the corresponding s, f_i, and g_i are shown in equation 3 for the above-mentioned separable approximations. The notation $\bar{z}$ is omitted for simplicity. The split-step Fourier method is:

$$s = 1, \quad f_1(x, y) = \exp\left\{i\left(\frac{\omega}{c(x, y)} - \frac{\omega}{c_0}\right)\Delta z\right\},$$
$$g_1(k_x, k_y) = \exp\left\{i\sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2}\Delta z\right\}.$$

The phase-screen method is:

$$s = 1, \quad f_1(x, y) = \exp\left\{\frac{ic_0\omega}{2}\Delta c(x, y)\Delta z\right\},$$
$$g_1(k_x, k_y) = \exp\left\{i\sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2}\Delta z\right\}.$$

The generalized-screen method (n*th* order) is:

$$s = n + 1,$$
$$f_1(x, y) = \exp\left\{i\left(\frac{\omega}{c(x, y)} - \frac{\omega}{c_0}\right)\Delta z\right\},$$
$$g_1(k_x, k_y) = \exp\left\{i\sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2}\Delta z\right\},$$
$$f_{j+1}(x, y) = i\omega\Delta z a_j \times \exp\left\{i\left(\frac{\omega}{c(x, y)} - \frac{\omega}{c_0}\right)\Delta z\right\}(\Delta c(x, y))^j,$$
$$g_{j+1}(k_x, k_y) = \exp\left\{i\sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2}\Delta z\right\} \times \left[\left(\sqrt{\frac{1}{c_0^2} - \frac{k_x^2}{\omega^2} - \frac{k_y^2}{\omega^2}}\right)^{-(2j-1)} - \left(\frac{1}{c_0}\right)^{-(2j-1)}\right],$$

where $j = 1, 2, \ldots, n$ and

$$a_1 = \frac{1}{2}, \quad a_j = (-1)^{j+1}\frac{1\cdot 3\cdots(2j-3)}{j!\,2^j}, \quad j \geq 2.$$

In the local framework, separable approximations based on the Chebyshev expansions are worthy of investigation (Halpern and Trefethen, 1988).

Global framework for constructing separable approximations

The global framework for constructing separable approximations consists of approximating the one-way operator (2) in a global interval by means of optimization, and it was developed by Chen and Liu (2004). We give a brief introduction to this method below.

We introduce variables $u = \omega/c(x, y, \bar{z})$ and $k = \sqrt{k_x^2 + k_y^2}$, and with these variables, the one-way operator (2) becomes

$$\mathcal{A}(u, k) = \exp\left(i\sqrt{u^2 - k^2}\Delta z\right). \quad (5)$$

The optimal separable approximation for equation 5 is to find functions $\phi(u)$, $\psi(k)$, and a complex number λ such that

$$\|\mathcal{A}(u, k) - \lambda\phi(u)\psi(k)^*\|_{L^2} = \min_{\tilde{\phi},\tilde{\psi},\tilde{\lambda}}\|\mathcal{A}(u, k) - \tilde{\lambda}\tilde{\phi}(u)\tilde{\psi}(k)^*\|_{L^2}, \quad (6)$$

where * denotes the complex conjugate, $\tilde{\lambda} \in \mathbb{C}$, and

$$\tilde{\phi} \in \left\{\tilde{\phi}(u) : \tilde{\phi}(u) \in L^2[a, b], \|\tilde{\phi}(u)\|_{L^2} = 1\right\},$$
$$\tilde{\psi} \in \left\{\tilde{\psi}(k) : \tilde{\psi}(k) \in L^2[c, d], \|\tilde{\psi}(k)\|_{L^2} = 1\right\}.$$

Here, $L^2[a, b]$ stands for the space consisting of square integrable functions defined on $[a, b]$. The norm $\|\cdot\|_{L^2}$ is defined by

$$\|f(x)\|_{L^2} = \left(\int_a^b |f(x)|^2 dx\right)^{1/2}, \quad \forall f(x) \in L^2[a, b].$$

Using a Lagrange multiplier, it can be proved easily that the solution to equation 6 is the eigenfunction corresponding to the eigenvalue with maximum modulus of the following linear integral equation system:

$$\begin{aligned}\int_c^d \mathcal{A}(u,k)\psi(k)dk &= \lambda\phi(u),\\ \int_a^b \mathcal{A}(u,k)^*\phi(u)du &= \lambda^*\psi(k).\end{aligned} \tag{7}$$

In general, the analytical solution of system (7) is not available and can be solved only numerically. Consider partitions of intervals $[a,b]$ and $[c,d]$ with nodes:

$$u_i = a + (i-1)\Delta u,\quad i = 1,2,\ldots,m+1;\quad \Delta u = \frac{b-a}{m},$$

$$k_j = c + (j-1)\Delta k,\quad j = 1,2,\ldots,n+1;\quad \Delta k = \frac{d-c}{n}.$$

Set $\phi = (\phi_1, \phi_2, \ldots, \phi_m)^T$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$, where $\phi_s = \phi(u_s), s = 1,2,\ldots,m$ and $\psi_q = \psi(k_q), q = 1,2,\ldots,n$. Let $A = (a_{i,j})$ be a matrix with entries:

$$a_{i,j} = \mathcal{A}(u_i, k_j),\quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n.$$

By solving the system (7) numerically, we can conclude that ϕ and ψ are the left and right singular vectors of A corresponding to the maximum singular value λ_1, respectively. For details, see Chen and Liu (2004). Let $\phi^{(1)}(u)$ and $\psi^{(1)}(k)$ denote the interpolation function of ϕ and ψ respectively.

Next, we obtain the optimal approximation with separable variables for $\mathcal{A}(u,k)$ as:

$$\mathcal{A}(u,k) \simeq \lambda_1\phi^{(1)}(u)\psi^{(1)}(k)^*.$$

To increase accuracy, set

$$\mathcal{A}_1(u,k) = \mathcal{A}(u,k) - \lambda_1\phi^{(1)}(u)\psi^{(1)}(k)^*.$$

We can obtain the optimal approximation with separable variables for $\mathcal{A}_1(u,k)$ by using the same method as used for $\mathcal{A}(u,k)$:

$$\mathcal{A}_1(u,k) \simeq \lambda_2\phi^{(2)}(u)\psi^{(2)}(k)^*.$$

Thus, we have the second-order approximation

$$\mathcal{A}(u,k) \simeq \lambda_1\phi^{(1)}(u)\psi^{(1)}(k)^* + \lambda_2\phi^{(2)}(u)\psi^{(2)}(k)^*.$$

Repeating this process, we finally obtain

$$\mathcal{A}(u,k) \simeq \sum_{l=1}^{t} \lambda_l\phi^{(l)}(u)\psi^{(l)}(k)^*, \tag{8}$$

where $t \le r$ and r is the rank of A.

The corresponding s, f_i, and g_i in equation 3 for the separable approximation (8) are as follows:

$$s = t,\quad f_l(x,y) = \lambda_l\phi^{(l)}(u),$$

$$g_l(k_x,k_y) = \psi^{(l)}(k)^*,\quad l = 1,2,\ldots,t.$$

NUMERICAL OPERATOR COMPARISONS

We perform some numerical operator comparisons in this section. For simplicity, consider here the 2D case [(x,z) section]. In the following, we will use the sign GS(n-1) to stand for the (n-1)*th*-order generalized-screen method and the sign OSAn for the n*th*-order optimal separable approximation. Notice that the letter n in these signs can be replaced by numbers. For example, OSA4 stands for the fourth-order optimal separable approximation. Because GS(n-1) and OSAn have the same computational complexity $(n+1)N_xN_ylog_2(N_xN_y)$, we will make comparisons between them.

For the velocity and frequency, we take a typical marine processing model: $f \in [0\,hz, 40\,hz]$ and $v \in [1500$ m/s, 4500 m/s$]$. The range of k_x is from $-2\pi/125$ to $2\pi/125$. Set $\Delta z = 10$ m. Further, we take

$$\Delta u = \frac{1}{40}\left[\frac{40\pi}{1500} - \frac{40\pi}{4500}\right];\quad \Delta k_x = \frac{1}{100}\left(\frac{4\pi}{125}\right).$$

In Figure 1, we take $\omega = 40\pi$ and show the error of GS(n-1) and OSAn for $n = 1,2,3,4$. The error is defined by

$$(\text{Error}(u))^2 = \frac{1}{T}\sum_{j=-50}^{50}\left|\mathcal{A}(u, j\Delta k_x) - \sum_{l=1}^{n} f_l(u)h_l(j\Delta k_x)\right|^2, \tag{9}$$

where $T = \sum_{j=-50}^{50}|\mathcal{A}(u, j\Delta k_x)|^2$.

The error in equation 9 is defined as a function of velocity and is the error for an arbitrarily heterogeneous model with a corresponding range of velocities. In numerical comparisons; however, it is customary to make comparisons for individual velocities. For example, see le Rousseau and de Hoop (2001), and we will follow this convention. The calculated errors reflect the difference between the calculated image and the true image. Under the same conditions (such as the same trace spacing, time sampling, and quality of recorded data), a smaller error means more accurate imaging. We sum over the wavenumber because in seismic migration the wavefield continuation usually is performed for individual frequencies.

For the generalized-screen method GS(n-1), we take the reference velocity as $c_0 = 1400$ m/s to avoid the branch point. For other reference velocity that is smaller than 1500 m/s, we can obtain similar results. For n = 1, the GS0 is merely the split-step Fourier method. From Figure 1, we see that the

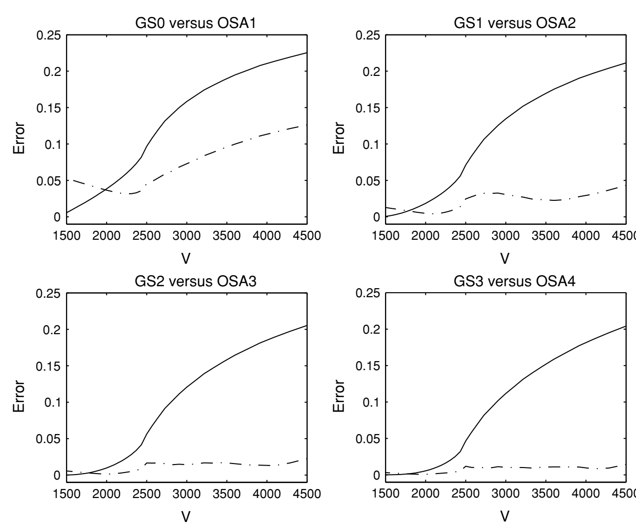

Figure 1. Comparison of the errors of GS(n-1) and OSAn for $n = 1,2,3,4$. GS(n-1): solid line; OSAn: dashed-dotted line. We take $\omega = 40\pi$.

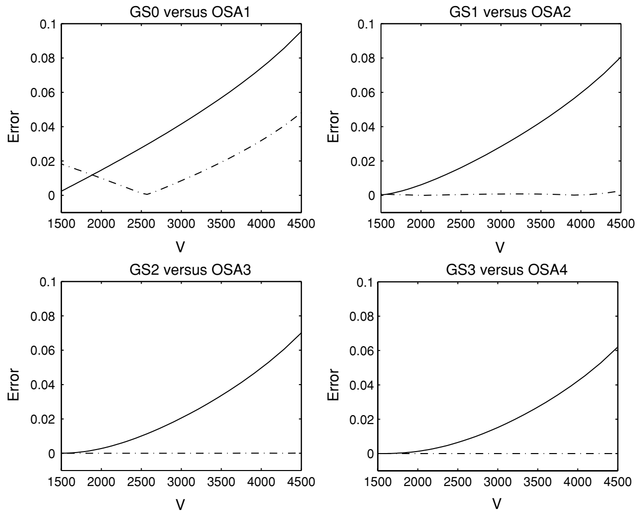

Figure 2. Comparison of the errors of GS(n-1) and OSAn for $n = 1, 2, 3, 4$. GS(n-1): solid line; OSAn: dashed-dotted line. We take $\omega = 80\pi$.

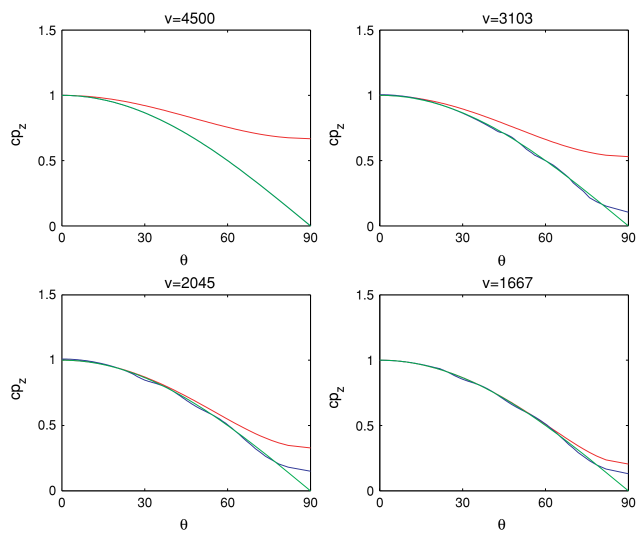

Figure 3. The phase curves of GS3, OSA4 and the exact one-way operator for different velocities are shown. Exact: green line; GS3: red line; OSA4: blue line. We take $\omega = 40\pi$.

error of GS(n-1) increases with increasing velocity. For large velocities, the error becomes large, and this situation does not significantly improve when the order n increases. The OSAn, on the other hand, has small errors for the whole velocity interval, and the error decreases rapidly when the order n increases. For other frequencies, we can obtain similar observations. Figure 2 shows the results for the frequency $\omega = 80\pi$.

In Figures 3 and 4, we compare the phase curves cp_z of GS3, OSA4 and the exact one-way operator for different velocities. Here, p_z is vertical slowness. The phase curve cp_z is defined by

$$cp_z = \sqrt{1 - \left(\frac{ck_x}{\omega}\right)^2} = \sqrt{1 - \sin^2\theta},$$

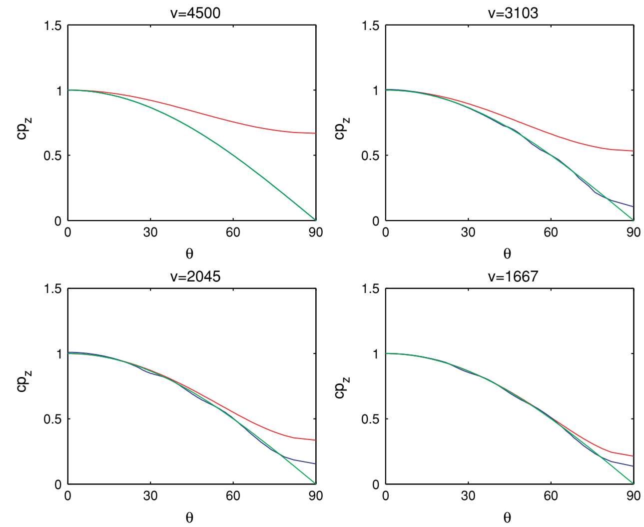

Figure 4. The phase curves of GS3, OSA4 and the exact one-way operator for different velocities are shown. Exact: green line; GS3: red line; OSA4: blue line. We take $\omega = 80\pi$.

where $\theta = \sin^{-1}(ck_x/\omega)$ is the propagation angle and the range of cp_z is normalized between 0 and 1. In the following figures, we use degree as the unit of θ.

We first take $\omega = 40\pi$. From Figure 3, we see that the phase curves of OSA4 and the exact one-way operator are in agreement for all velocities. Again, the error in the phase curves of GS3 increases with increasing velocity. This is caused by the choice of the reference velocity. For other frequencies, we can draw the same conclusions. Figure 4 shows the results for $\omega = 80\pi$.

CONCLUSIONS

In this short note, we have explored the global approximation properties of the optimal separable approximations and compared them with the local-based methods such as the generalized-screen method. Based on this research, we introduce the concept of separable approximations for the one-way wave operator. Two kinds of separable approximations based on the local and global frameworks are compared. We perform numerical operator comparisons in which frequency and velocity ranges are taken from a typical marine processing model. The results show that for the same computational complexity (similar CPU time), the global method is accurate for all the velocities, whereas the error in the local method (generalized-screen method) increases with increasing velocity. Both methods can be applied in arbitrarily complex media; i.e., there is no restriction whatsoever to somehow well-behaved models, such as weak contrasts, smooth, moderately dipping, etc. However, the OSA method, different from the GS method, does not rely on a background medium and has much faster convergence.

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