Iterative implementation of the adaptive regularization yields optimality

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Abstract The adaptive regularization method is first proposed by Ryzhikov et al. for the deconvolution in elimination of multiples. This method is stronger than the Tikhonov regularization in the sense that it is adaptive, i.e. it eliminates the small eigenvalues of the adjoint operator when it is nearly singular. We will show in this paper that the adaptive regularization can be implemented iterately. Some properties of the proposed non-stationary iterated adaptive regularization method are analyzed. The rate of convergence for inexact data is proved. Therefore the iterative implementation of the adaptive regularization can yield optimality.

Keywords: ill-posed problems, non-stationary iterated adaptive regularization, optimality.

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1 Introduction

In this paper we investigate the approximate solution of the ill-posed operator equation

$$Tx = y, \tag{1}$$

where T is a bounded linear operator from Hilbert space X to the Hilbert space Y. Assume that we are interested in the MNS-solution (minimum-norm least-squares solution) x^+ of (1). Then it is well-known that

$$x^+ = T^+ y, \tag{2}$$

where T^+ denotes the Moore-Penrose generalized inverse of T. For the nonclosed range $R(T)$ of T, the MNS-solution x^+ exists only for

$$D(T^+) = R(T) + R(T)^\perp \subset Y$$

and depends discontinuously on the right-hand side. A prototype for such an ill-posed problem is the Fredholm integral equations of the first kind

$$(Tx)(t) := \int_0^1 k(s,t)x(t)dt = y(s), \ \ s \in [0,1],$$

where k is a nondegenerate L^2-kernel and $X = Y = L^2[0,1]$ (see refs. [1—3]).

In many applications, the right-hand side y cannot be obtained exactly. In fact only a perturbed version $y_\delta \in Y$ is available, which satisfies

$$\|y_\delta - y\| \leqslant \delta, \tag{3}$$

where δ is an $a\ priori$ known/estimated error level. Since T^+ is generally unbounded, T^+y_δ is not a reasonable approximation to T^+y, even if it exists. Therefore, one has to use the so-called regularization method to approximate T^+y.

A well established regularization method for the solution of ill-posed inverse problems is the Tikhonov regularization[2]. Instead of solving an MNS-solution of (1), this method minimizes a regularized functional

$$M^\alpha[x, y_\delta] \stackrel{def}{=} \frac{1}{2}\|Tx - y_\delta\|^2 + \frac{\alpha}{2}\|Lx\|^2, \tag{4}$$

where $\alpha > 0$ is the so-called regularization parameter, L is a scale operator. In this paper, we choose $L \equiv I$ (the identity). For $\alpha > 0$, we denote by x_δ^α the minimizer of (4). By the first-order necessary condition, x_δ^α is the unique solution of the Euler equation

$$(T^*T + \alpha I)x = T^*y_\delta. \tag{5}$$

For the theory of Tikhonov regularization, we refer to refs. [3, 4]. It is well-known that if the regularization parameter α is chosen to be dependent on δ such that

$$\lim_{\delta \longrightarrow 0} \frac{\delta^2}{\alpha(\delta)} = 0 \text{ and } \lim_{\delta \longrightarrow 0} \alpha(\delta) = 0$$

(see ref. [2]), then $\lim_{\delta \longrightarrow 0} \|x_\delta^{\alpha(\delta)} - T^+y\| = 0$. Moreover, if the exact solution fulfills the smoothness property

$$T^+y \in R((T^*T)^\nu) \tag{6}$$

for some $\nu \in (0,1]$, then for an $a\ priori$ choice of α such that

$$\alpha(\delta) = c\delta^{\frac{2}{2\nu+1}},\ c > 0,$$

one obtains the convergence rate[4,5]

$$\|x_\delta^{\alpha(\delta)} - T^+y\| = O(\delta^{\frac{2\nu}{2\nu+1}}).$$

Note that the solution of (5) is in a direct way. The standard Tikhonov regularization can be implemented iteratively[6−9]. Hanke et al.[6] proposed the iteration process in a non-stationary way. Nice properties were obtained due to such a process. Schock[10] considered the regularization with adjoint operators. Recently Ryzhikov et al.[11] and Wang et al.[12,13] considered the adaptive regularization method (simply denoted by AR) for solving ill-posed inverse problems (1) with the adjoint operator. Ryzhikov et al. successfully utilized this kind of methods to solve geoscience deconvolution problems in elimination of multiples. Denoting $H = T^*T$, which is an adjoint operator, Ryzhikov et al.'s adaptive regularization is based on the minimization problem

$$\min J^\alpha[x, y_\delta] \stackrel{def}{=} \frac{1}{2}\|Tx - y_\delta\|^2 + \frac{\alpha}{2}\|x\|_D^2, \tag{7}$$

where $\|x\|_D \overset{def}{=} \sqrt{(Dx, x)}$, D is a definite or semi-definite operator. Clearly the minimization of (4) is equivalent to (7) if $L = D^{\frac{1}{2}}$ and D is symmetric. Now choosing $D = H^{-1}$ and denoting by x_δ^α the minimizer of (7), for $\alpha > 0$, Ryzhikov et al. obtained

$$(H^2 + \alpha I)x_\delta^\alpha = Hz_\delta, \tag{8}$$

where $z_\delta = T^* y_\delta$. The filtering function of the AR is defined by $R^{\text{AR}}(\lambda) = \lambda(\lambda^2 + \alpha)^{-1}$. The remarkable difference between the adaptive regularization and the Tikhonov regularization is that the former can simultaneously eliminate the null space elements of the operator when it approaches singularity while the latter needs the regularization parameter to suppress the singularity[11,12].

We observe that the adaptive regularization can be implemented iteratively. By noting that the regularization parameter α should be variational in each iteration to make a trade-off of the ill-posedness of the problem, we prefer to choose the parameter in a geometric way[2,6]. Thus the iteration process can be generated as follows:

$$(H^2 + \alpha_n I)x_n^\delta = \alpha_n x_{n-1}^\delta + Hz_\delta. \tag{9}$$

The geometric choice of regularization parameters α is in the form

$$\alpha_n = \alpha_0 \xi^{n-1}, \quad \xi \in (0, 1). \tag{10}$$

Since the regularization parameter is variational in each iteration, we call this iteration process a non-stationary iterated adaptive regularization (NSIAR) method. In the following sections, we will give a theoretical analysis of the method. This paper is organized as follows: in sec. 2, the convergence properties of the method are presented when the exact data are given; in sec. 3, the rate of convergence of the method is proved when the perturbed data are obtained, hence the NSIAR can approach asymptotic optimality.

2 Convergence properties of the NSIAR for exact data

We begin from the noise-free data. In this case, the iterated version of the adaptive regularization is generated as follows:

$$(H^2 + \alpha_n I)x_n = \alpha_n x_{n-1} + Hz \tag{11}$$

or

$$x_n = \alpha_n (H^2 + \alpha_n I)^{-1} x_{n-1} + (H^2 + \alpha_n I)^{-1} Hz, \tag{12}$$

where $z = T^* y$. For simplicity, we choose the initial guess value $x_0 = 0$. Also without loss of generality, we assume that $\|T\| \leqslant 1$ (otherwise, a constant multiple $\frac{1}{\|T\|}$ can be performed on both sides of (1)), hence $\|H\| \leqslant 1$.

Note that both the operator $(H^2 + \alpha_n I)^{-1}$ and the operator $(H^2 + \alpha_n I)^{-1} H$ are everywhere defined and bounded with $\|(H^2 + \alpha_n I)^{-1}\| \leqslant \frac{1}{\alpha_n}$ and $\|(H^2 + \alpha_n I)^{-1} H\| \leqslant \frac{1}{1+\alpha_n}$. Therefore, for each fixed n, the sequence $\{x_n\}$ generated by (11) or (12) is stable with respect to perturbations in y.

We begin by noting that $y = Tx^+$, so

$$(H^2 + \alpha_n I)^{-1} Hz = (H^2 + \alpha_n I)^{-1} H^2 x^+$$
$$= [I - \alpha_n (H^2 + \alpha_n I)^{-1}]x^+ = x^+ - \alpha_n (H^2 + \alpha_n I)^{-1} x^+.$$

From (12) and by simple calculation, we obtain

$$x^+ - x_n = \alpha_n (H^2 + \alpha_n I)^{-1}(x^+ - x_{n-1})$$
$$= \cdots = [\Pi_{i=1}^n \alpha_i (H^2 + \alpha_i I)^{-1}]x^+.$$

Thus x_n can be expressed as

$$x_n = R^{\mathrm{NSIAR}}(H)x^+, \tag{13}$$

where $R^{\mathrm{NSIAR}}(\lambda) \stackrel{def}{=} 1 - \Pi_{i=1}^n \frac{\alpha_i}{\lambda^2+\alpha_i}$ and $\lambda \in [0, 1]$.

It is clear that $R^{\mathrm{NSIAR}}(\lambda) \to 1$ as $\alpha_i \to 0,\ i \to \infty,\ n \to \infty$. Therefore, $x_n \to x^+$ as $i \to \infty,\ n \to \infty$. In addition, the remarkable feature of the NSIAR is $R^{\mathrm{NSIAR}}(\lambda) \to 0$ as $\lambda \to 0$, which means that if the operator $H = T^*T$ is degenerated and has an eigenvalue being null, then the NSIAR-inverse operator eliminates null-space components at all, whereas for Tikhonov regularization, if the operator $H = T^*T$ has an eigenvalue being null, then the filtering function $R^{Tikh}(\lambda) = (\lambda + \alpha)^{-1} \to \alpha^{-1}$ as $\lambda \to 0$. Thus the NSIAR is more adaptive.

If we denote $r_n(\lambda) = \Pi_{i=1}^n \frac{\alpha_i}{\lambda^2+\alpha_i}$, then $R^{\mathrm{NSIAR}}(\lambda) = 1 - r_n(\lambda)$. Thus we have

$$x^+ - x_n = r_n(H)x^+.$$

Furthermore, if $x^+ = H^\nu \omega$ for some $\omega \in D(H^\nu)$ and $\nu > 0$, i.e. x^+ satisfies the so-called source condition[1,5], then

$$x^+ - x_n = S_{n,\nu}(H)\omega, \tag{14}$$

where $S_{n,\nu}(\lambda) := \lambda^\nu r_n(\lambda)$.

The function

$$S_{n,\nu}(\lambda) \stackrel{\mathrm{def}}{=} \lambda^\nu \Pi_{i=1}^n \frac{\alpha_i}{\lambda^2 + \alpha_i},\ \lambda \in [0, 1]$$

plays an important role in the analysis of the errors between the exact solution x^+ and the iterates x_n, where $\nu > 0,\ \alpha_i > 0$ are given parameters. As we are interested in fixed $\nu > 0$ and $n \to \infty$, we shall assume $n > \frac{\nu}{2}$ (note that for $n \leqslant \frac{\nu}{2}$, $S_{n,\nu}(\lambda)$ is increasing in λ). It is easy to find that

$$S_{n,\nu}(\lambda) \leqslant \lambda^\nu \frac{\alpha_i}{\lambda^2 + \alpha_i} \leqslant \lambda^{\frac{\nu}{n}} \frac{\alpha_i}{\lambda^2 + \alpha_i} \stackrel{def}{=} f_\nu(\lambda). \tag{15}$$

We call $f_\nu(\lambda)$ the dominant function. An easy calculation shows that

$$df_\nu(\lambda)/d\lambda = 0$$

if and only if

$$\lambda = \left(\frac{\alpha_i \nu}{2n - \nu}\right)^{1/2} \quad \text{as } 0 < \nu < 2n$$

or $\lambda = 0 \quad$ as $\nu > n$.

But clearly, $f_\nu(0) = 0$. So we let $\lambda^* = (\frac{\alpha_i \nu}{2n-\nu})^{1/2}$. We can easily deduce that

$$f'_\nu(\lambda) = \lambda^{\frac{\nu}{n}-1} \frac{\alpha_i}{\lambda^2 + \alpha_i} \left(\frac{\nu}{n} - 2\frac{\lambda^2}{\lambda^2 + \alpha_i} \right)$$

$$f'_\nu(\lambda) > 0 \text{ if and only if } \lambda \in (0, \lambda^*),$$

$$f'_\nu(\lambda) < 0 \text{ if and only if } \lambda \in (\lambda^*, 1).$$

This means λ^* is the maximal value point. Thus we obtain

Lemma 2.1. If $0 < \nu < 2n$, then $\max_{\lambda\in[0,1]} S_{n,\nu}(\lambda) \leqslant C_{n,\nu}\alpha_i^{\nu/2n}$ for all $i = 1, 2, \cdots, n$, where $C_{n,\nu} = (1 - \frac{\nu}{2n})\Big(\frac{\frac{\nu}{2n}}{1-\frac{\nu}{2n}}\Big)^{\frac{\nu}{2n}}$ is upper bounded.

Proof. Since $S_{n,\nu}(\lambda) \leqslant f_\nu(\lambda)$ and

$$\max f_\nu(\lambda) = f_\nu\left(\left(\frac{\alpha_i \nu}{2n - \nu} \right)^{1/2} \right) = \left(1 - \frac{\nu}{2n} \right) \left(\frac{\frac{\nu}{2n}}{1 - \frac{\nu}{2n}} \right)^{\frac{\nu}{2n}} \alpha_i^{\frac{\nu}{2n}},$$

the result is clear. Now we prove the boundedness of $C_{n,\nu}$. Denoting $u = \frac{\nu}{2n}$, we have the following continuous function:

$$C(u) = (1-u)\Big(\frac{u}{1-u}\Big)^u, \quad 0 < u < 1.$$

By calculus, we find that $u = \frac{1}{2}$ is the only stationary point and $C(\frac{1}{2}) = \frac{1}{2}$. Moreover, $C(u)_{u\to 0} = C(u)_{u\to 1} = 1$. So $C_{\max}(u)_{u\in(0,1)} < 1$, hence the boundedness of $C_{n,\nu}$ is concluded. □

To analyze the convergence order optimality of the regularization method, people usually assume some kinds of source conditions are satisfied[1,5], i.e. assume $x^+ = H^\nu \omega$ for some $\omega \in D(H^\nu)$ and fix $\nu > 0$. Under the source condition, we have

Theorem 2.2. If $x^+ = H^\nu\omega$ for some $\nu > 0$ and $\omega \in D(H^\nu)$, then for all α_i $(i = 1, 2, \cdots, n)$,

$$\|x^+ - x_n\| \leqslant C_{n,\nu}\alpha_i^{\frac{\nu}{2n}}\|\omega\|. \tag{16}$$

Proof. From (14) and Lemma 2.1,

$$\|x^+ - x_n\| \leqslant \|S_{n,\nu}(H)\|\|\omega\| \leqslant C_{n,\nu}\alpha_i^{\frac{\nu}{2n}}\|\omega\|.$$ □

Lemma 2.1 and Theorem 2.2 hold true for all α_i $(i = 1, 2, \cdots, n)$. For fixed ν and ξ, $\xi \in (0, 1)$, $\alpha_i = \alpha_0\xi^{i-1}$ decreases quickly as i increases. Therefore from (16), we concluded that the NSIAR converges quickly.

3 Rate of convergence for perturbed data

Assume that y_δ is an approximation to the exact data y with error level δ such that $\|y - y_\delta\| \leqslant \delta$. Let $\{x_n^\delta\}$ be the sequence generated by (12) with the data y_δ, i.e.

$$x_n^\delta = \alpha_n(H^2 + \alpha_n I)^{-1}x_{n-1}^\delta + (H^2 + \alpha_n I)^{-1}Hz_\delta, \tag{17}$$

where $z_\delta = T^* y_\delta$.

From (12) we see that x_n can be expressed as

$$x_n = q_n(H)z,\ x_0 := 0,$$

where $q_n(\lambda)$ is generated in the form

$$q_n(\lambda) = (\lambda^2 + \alpha_n)^{-1}(\alpha_n q_{n-1}(\lambda) + \lambda),\ q_0(\lambda) := 0.$$

Similarly, for the perturbed data y_δ, we have the expression

$$x_n^\delta = q_n(H)z_\delta,\ x_0^\delta := 0,$$

where $q_n(\cdot)$ is defined as above. It follows that

$$1 - \lambda q_n(\lambda) = \frac{\alpha_n}{\lambda^2 + \alpha_n}(1 - \lambda q_{n-1}(\lambda)),$$

that is, $q_n(\lambda) = \frac{1-r_n(\lambda)}{\lambda}$ with $r_n(\lambda) = \Pi_{i=1}^n \frac{\alpha_i}{\lambda^2+\alpha_i}$. We have the following estimation:

$$0 \leqslant q_n(\lambda) \leqslant \lambda^{-1}.$$

Noticing the expression for x_n and x_n^δ by $q_n(\cdot)\cdot$ and $x_n,\ x_n^\delta \in D(T)$, we have

$$\begin{aligned} \|x_n - x_n^\delta\|^2 &= \|q_n(H)(z - z_\delta)\|^2 \\ &= (q_n(H)(z - z_\delta), q_n(H)(z - z_\delta)) \\ &= (q_n(H)T^*(y - y_\delta), q_n(H)T^*(y - y_\delta)) \\ &\leqslant \delta^2\|T\|^{-2}, \end{aligned}$$

i.e.

$$\|x_n - x_n^\delta\| \leqslant \delta\|T\|^{-1}.$$

Hence

$$\|Tx_n - Tx_n^\delta\| \leqslant \|T\|\|x_n - x_n^\delta\| \leqslant \delta.$$

To deeply investigate the relation between the error level δ and the regularization parameter α_i, we need the following estimation (by triangular inequality) for $\lambda \geqslant 0$:

$$0 \leqslant q_n(\lambda) \leqslant \frac{\lambda^2 + 2\alpha_i}{\lambda^3 + \alpha_i\lambda} \leqslant \frac{3}{2\sqrt{\alpha_i}}.$$

Thus we obtain

$$\begin{aligned} \|x_n - x_n^\delta\|^2 &= (x_n - x_n^\delta, x_n - x_n^\delta) \\ &= (q_n(H)T^*(y - y_\delta), q_n(H)T^*(y - y_\delta)) \\ &\leqslant \|q_n(H)q_n(H)H\|\delta^2 \\ &\leqslant \|q_n(H)\|\delta^2 \\ &\leqslant \frac{3}{2\sqrt{\alpha_i}}\delta^2 \end{aligned}$$

for any $\alpha_i,\ i = 1, 2, \cdots, n$. We state the above results as the following lemma:

Lemma 3.1. For the sequence $\{x_n\},\ \{x_n^\delta\}$, we have the following stability results:

$$\|x_n - x_n^\delta\| \leqslant \frac{\sqrt{3/2}}{\sqrt[4]{\alpha_i}}\delta,\ \ i = 1, 2, \cdots, n$$

and

$$\|Tx_n - Tx_n^\delta\| \leqslant \delta.$$

Theorem 3.2. If $\alpha_n \to 0$ and $\frac{\delta^4}{\alpha_n} \to 0$ as $\delta \to 0$ and $n \to \infty$, then x_n^δ defined by (17) converges to $x^+ = T^+ y$.

Proof. By (13) and Lemma 3.1, the results can be easily obtained. □

From Theorem 3.2 we know that the iteration formula (17) converges as long as the regularization parameters α_n defined by (10) do not exceed $\delta^k,\ k < 4$. To emphasize α_n is related to δ, sometimes we also write α_n as $\alpha_n(\delta)$. However we must bear in mind that the regularization parameter α_n decreases as the iteration proceeds. Therefore we conclude that the iterations cannot proceed infinitely, in other words, the iterations should be terminated in finite steps.

In the following, we turn to investigate the regularity of NSIAR for the noisy data y_δ. We will show that the iterative implementation of the adaptive regularization can yield optimality. To prove the results, we need the following lemma.

Lemma 3.3. Let $x^+ \neq 0$. Then for all $\delta > 0$ there exists a unique $\alpha := \alpha_l(\delta)$ for some $l \in N$ satisfying the equation

$$\|x_n - x^+\| = \delta / \sqrt[4]{\alpha_l}. \tag{18}$$

Proof. Since $Tx^+ = Qy$, where Q denotes the orthogonal projector of Y onto $\overline{R(T)}$, from (13) we have

$$x_n - x^+ = -\Pi_{i=1}^n \alpha_i (H^2 + \alpha_i I)^{-1} x^+.$$

Denoting the spectral family of the operator $H = T^*T$ as E_λ, we have

$$\sqrt{\alpha}\|x_n - x^+\|^2 = \int_0^1 \Pi_{i=1}^n \frac{\sqrt{\alpha}\alpha_i^2}{(\lambda^2 + \alpha_i)^2} d\|E_\lambda x^+\|^2.$$

Hence if we define

$$\phi(\alpha) = \int_0^1 \Pi_{i=1}^n \frac{\sqrt{\alpha}\alpha_i^2}{(\lambda^2 + \alpha_i)^2} d\|E_\lambda x^+\|^2 - \delta^2,$$

then by definition $x^+ \in N(T)^\perp$ and since $x^+ \neq 0$, we know that $\phi(\alpha)$ is continuous and strictly and monotonically increasing with

$$\lim_{\alpha \to 0} \phi(\alpha) = -\delta^2 \quad \text{and} \quad \lim_{\alpha \to \infty} \phi(\alpha) = \infty,$$

which proves the lemma. □

Lemma 3.3 together with Theorem 2.2 indicates that $\alpha := \alpha_l(\delta)$ is continuous and strictly and monotonically increasing with $\alpha \to 0$ as $\delta \to 0$ and $\alpha \to \infty$ as $\delta \to \infty$. Now we prove the regularity results. We find that the rate of convergence of NSIAR can approach asymptotic optimality.

Theorem 3.4. Let $x^+ \in R(H^\nu)$ with some $\nu > 0$. Then $\|x_n^\delta - x^+\| = O(\delta^{\frac{4\nu}{4\nu+1}})$.

Proof. By Lemmas 3.1, 3.3 and the triangle inequality, we have

$$\begin{aligned} \|x_{n(\delta)}^\delta - x^+\| &\leqslant \|x_n^\delta - x_n\| + \|x_n - x^+\| \\ &= \frac{\sqrt{3/2}}{\sqrt[4]{\alpha_i}}\delta + \|x_n - x^+\|. \end{aligned} \tag{19}$$

Now let α_l be as in Lemma 3.3 and define

$$D_n = \frac{\|x_n - x^+\|}{\alpha_l^\nu}.$$

Then

$$\alpha_l^{4\nu} D_n^4 = \delta^4/\alpha_l.$$

By Theorem 2.2, we conclude that D_n is upper bounded. Thus α_l can be written as

$$\alpha_l = (\delta D_n^{-1})^{\frac{4}{4\nu+1}}.$$

Note that the first part of (19) is valid for any $\alpha_i,\ i = 1, 2, \cdots$. In particular, we choose $\alpha_i = \alpha_l$ and have

$$\begin{aligned}\|x_n^\delta - x^+\| &\leqslant \frac{\sqrt{3/2}}{\sqrt[4]{\alpha_i}}\delta + \|x_n - x^+\| = O\left(\frac{\delta}{\sqrt[4]{\alpha_l}}\right) \\ &= O\left(\delta^{\frac{4\nu}{4\nu+1}} D_n^{\frac{1}{4\nu+1}}\right) = O(\delta^{\frac{4\nu}{4\nu+1}}).\end{aligned}$$

□

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References

1. Engl, H. W., Hanke, M., Neubauer, A. et al., Regularization of Inverse Problems, Dordrecht: Kluwer Academic, 1996.
2. Tikhonov, A. N., Arsenin,V. Y. et al., Solutions of Ill-Posed Problems, New York: Wiley, 1977.
3. Xiao, T. Y., Yu, S. G., Wang, Y. F. et al., Numerical Methods for Inverse Problems, Beijing: Science Press, 2003.
4. Groetsch, C. W., The Theory of Tikhonov Regularization for Fredholm Equations of The First Kind, Boston, MA: Pitman, 1984.
5. Tautenhahn, U., Optimality for ill-posed problems under general source conditions, Numer. Funct. Anal. Optim., 1998, 19: 377—398.
6. Hanke, M., Groetsch, C. W. et al., Nonstationary iterated Tikhonov regularization, Journal of Optimization Theory and Applications, 1998, 98: 37—53.
7. King, J. T., Chillingworth, D. et al., Approximation of generalized inverses by iterated regularization, Numer. Funct. Anal. Optim., 1979, 1: 499—513.
8. Gfrerer, H., An *a-posteriori* parameter choice for ordinary and iterated Tikhonov regularization of ill-posed problems leading to optimal convergence rates, Mathematics of Computation, 1987, 49: 507—522.
9. He, G. Q., Liu L. X. et al., A kind of implicit iterative methods for ill-posed operator equations, Journal of Computational Mathematics, 1999, 17: 275—284.
10. Schock, E., Regularization of ill-posed equations with self-adjoint operators, in German-Italian Symposium on the Applications of Mathematics in Technology (eds. Boffi, V., Neunzert, H.), Stuttgart: Teubner, 1984, 340—351.
11. Ryzhikov, G. A., Biryulina, M. S. et al., Sharp deconvolution in elimination of multiples, 68th Ann. Internat. Mtg. Soc. Expl. Geophys., 1998, 1983—1986.
12. Wang, Y. F., On the Optimization and Regularization Algorithms for Inverse Problems, Ph.D. Thesis, AMSS, Chinese Academy of Sciences, Beijing, China, 2002.
13. Wang, Y. F., Ma, Q. H. et al., Iterated adaptive regularization for the operator equations of the first kind, Advances in Applied Mathematics (Proceedings in SIAM Gator Conference), 2004, in press